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Renewal Representations for Markov Operators

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As is known, due to the existence of an embedded renewal structure, the iterates of a Harris recurrent Markov operator can be represented as a (delayed) renewal sequence. We show that these kind of representations also exist for a larger class of Markov operators, provided only that certain "filling schemes" are "successful." As applications of the theory we study the co-Feller operators and Markov operators which "contract the variation."

1. INTRODUCTION

A basic technique of analyzing a stochastic system is to find an "isomorphic" system which has some kind of independence structure. When this succeeds the classical methods concerned with independent random variables become available.

An example of this is provided by a recurrent Markov chain on a countable state space. There the idea is to fix one state and to split the sample path of the Markov chain into blocks. By a block is meant the (random) sequence of states between two consecutive visits to the fixed state. Because of the Markov property the blocks are independent and identically distributed. One can also say that the Markov chain *regenerates* at the visit epochs to the fixed state. All this belongs to standard material in textbooks on stochastic processes (see, e.g., [5]).

Later (see [1, 13]) it was observed that (under a φ-recurrence condition, see (1.1) below) this kind of regeneration occurs also in Markov chains defined on general measurable spaces. Stationary processes associated with one-dimensional Gibbs fields have recently been proved regenerative, too (see [2, 9]).

The main purpose of the present article is to show that embedded independence structures may exist also in other markovian systems (even deterministic) than the φ-recurrent Markov chains. Although we will have regeneration in a very weak sense, it is strong enough, however, to produce the main limit results and the existence of an "invariant element."

The famous isomorphism theorem by Friedman and Ornstein [6] states that the so called weak Bernoulli systems are isomorphic to systems of i.i.d.

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random variables (Bernoulli systems). The definitions, methods, and results of the present article are quite different in character from those in the paper by Friedman and Ornstein (cf. Section 6, Corollary 6.4).

In order to motivate the studies to follow we shall start by describing the renewal representation of φ-recurrent Markov chains (or Harris chains, as they are also called). As regards Markov chains we adopt the notation and terminology of [14] (see also [17, 18]).

Let $(X_n; n=0, 1, ...)$ be a *Harris chain* on a measure space $(E, \mathscr{E}, \varphi)$, that means a Markov chain satisfying the *φ-recurrence* condition

$$P\{X_n \in A \text{ i.o.} \mid X_0 = x\} = 1 \qquad \text{for all } x \in E, A \in \mathscr{E} \text{ with } \varphi(A) > 0. \quad (1.1)$$

It is clear that then the n-step transition probabilities

$$P^n(x, \cdot) := P\{X_n \in \cdot \mid X_0 = x\}, \qquad x \in E, n \geqslant 1,$$

cannot all be singular (w.r.t. the reference measure φ). In fact, the following highly non-trivial result holds true:

C-SET LEMMA [3, 4, 8, 15]. *For a Harris chain there exist an integer $m_0 \geqslant 1$, a constant $\beta > 0$, a set $C \in \mathscr{E}$ with $\varphi(C) > 0$ and a probability measure ν on $(E, \mathscr{E})$ such that*

$$P^{m_0}(x, \cdot) \geqslant \beta\nu(\cdot) \qquad \textit{for all } x \in E.$$

(*Note that, by* (1.1) *the set C is visited infinitely often by the Markov chain* (X_n).)

When $m_0 = 1$ in the above inequality, a "regeneration scheme" can be constructed on the basis of the C-set Lemma:

REGENERATION LEMMA [1, 13]. *Suppose that (X_n) is a Harris chain with $m_0 = 1$. Then there exists a sequence $0 \leqslant T(0) < T(1) < \cdots$ of random times such that (X_n) regenerates at these epochs, i.e., conditional on $T(i) = n$ (for some i) the Markov chain starts anew at the time n with the same distribution* $(=\nu)$.

Remark. In the case, where E is a countable set and $\alpha \in E$ is a recurrent state, we can take $T(0)$, $T(1)$, ... simply to be the successive time epochs n such that $X_n = \alpha$. Then we have $m_0 = 1$, $\beta = 1$, $C = \{\alpha\}$, and $\nu = P(\alpha, \cdot)$.

It follows from the Regeneration Lemma that the n-step transition probabilities $P^n(x, \cdot)$ can be represented as a "delayed" renewal sequence. Namely, for $n = 0, 1, ...$, let

$$Q_n(x,\cdot) := P\{X_n \in \cdot,\ T(0) > n \mid X_0 = x\},$$
$$a_n(x) := P\{T(0) = n \mid X_0 = x\},$$
$$u_n := P\{T(i) = n \text{ for some } i \mid T(0) = 0\},$$
$$v_n := P\{X_n \in \cdot,\ T(1) > n \mid T(0) = 0\}.$$

Then (u_n) is a renewal sequence satisfying the renewal equations

$$u_0 = 1, \qquad u_n = \sum_{m=1}^{n} b_m u_{n-m} \qquad \text{for} \quad n \geqslant 1,$$

where

$$b_m := P\{T(i+1) - T(i) = m\} = P\{T(1) = m \mid T(0) = 0\},$$

and we have

$$P^n(x,\cdot) := Q_n(x,\cdot) + \mathbf{a}(x) * \mathbf{u} * \mathbf{v}_n, \tag{1.2}$$

where the notation $\mathbf{a}(x) * \mathbf{u} * \mathbf{v}_n$ means the convolution $\sum_{i,j,k:i+j+k=n} a_i(x) u_j v_k$ of the sequences $\mathbf{a}(x) := (a_n(x))$, $\mathbf{u} := (u_n)$, and $\mathbf{v} := (v_n)$ (see also [14, Theorem 4.1(iv)]). Moreover, the measure

$$\pi(A) := \sum_{n=0}^{\infty} v_n(A) = E \operatorname{card}\{X_n \in A, 0 \leqslant n < T(1) \mid T(0) = 0\}, \qquad A \in \mathscr{E},$$

is invariant for P, that means

$$\pi = \pi P := \int \pi(dx)\, P(x,\cdot) \qquad \text{(see [14, Theorem 5.2])}.$$

We also have the "undelayed renewal representation"

$$vP^n = \mathbf{u} * v_n \qquad [14, \text{Theorem } 4.1(\text{iii})]. \tag{1.3}$$

(Note the slight difference in the terminology as compared with that of [14]. What we here call "undelayed" is "delayed" in [14], while the "delayed" representation (1.2) has no particular naming in [14].)

By integrating both sides of (1.2) w.r.t. an arbitrary probability measure λ on $(E, \mathscr{E})$ we obtain the delayed renewal representation corresponding to the initial distribution λ of the Markov chain,

$$\lambda P^n = \lambda_n + \mathbf{a}(\lambda) * \mathbf{u} * \mathbf{v}_n. \tag{1.4}$$

Here $\lambda_n := \lambda Q_n$ and $\mathbf{a}(\lambda)$ denotes the sequence $(\int a_n\, d\lambda)$.

As remarked above, the n-step transition probabilities of a Harris chain are necessarily non-singular for some n. Therefore many important Markov chains, among them the (deterministic) Markov chains generated by transformations on a general measure space and also the Markov chains associated with particle systems on integer lattices, cannot belong to the class of Harris chains (expect in very special cases). On the other hand the regeneration scheme described above was just based on the existence of absolutely continuous components in the transition probabilities. So it seems that the regeneration method, which makes the study of Harris chains essentially as easy as the study of recurrent Markov chains on countable state spaces, is no more available outside the class of Harris chains.

The main purpose of the present article is to show that, however, absolute continuity is not a necessary condition for renewal representations of the type (1.3) and (1.4) to exist. The key observation leading to this extension is to note that formulas of the type (1.3) and (1.4) make sense also in weaker structures than is the (natural) measure theoretic structure of Harris chains. (Note that in these formulas the elements νP^n, ν_n, λP^n, and λ_n are positive measures and the elements u_n and $a_n(\lambda)$ are positive scalars.) It is clear that the "minimal structure" where these formulas are well defined is the structure of an ordered vector space. (That means, νP^n, ν_n, λP^n, and λ_n are elements of an ordered vector space. This will be made precise in Sections 2 and 3.) In order to be able to take limits as $n \to \infty$ we have to impose also a lattice cone structure having some (minimal) completeness property.

As one basic new example we have in mind the so called co-Feller chains, i.e., Markov chains where the transition probability P maps an initial distribution λ with lower semicontinuous density (w.r.t. some reference measure μ on the underlying topological state space E) to a distribution with the same property. So we are naturally led to consider the Markov operator P as an operator on the cone of lower semicontinuous functions on a topological space.

The plan of this article is briefly as follows:

In Section 2 we introduce the basic concepts such as Markov operators on vector lattices and the special cases of co-Feller operators, Feller operators, and Markov operators which contract the variation.

In Section 3 we define the concept of a renewal representation. We shall also give a probabilistic characterization to it in terms of a recurrent Markov chain having a countable number of states.

In Section 4 we describe the main limit theorems which hold true for Markov operators with a renewal representation.

In Section 5 we formulate simple "filling schemes," which turn out to be equivalent to the renewal representation.

Section 6 is devoted to the study of the co-Feller operators. It turns out that for them the L_1-convergence towards the equilibrium (Orey's convergence theorem) is equivalent to the existence of a renewal representation. As an example we consider weakly Bernoulli shifts on a finite alphabet.

In treating some examples (notably the example of a Markov operator contracting the variation) we have to generalize the representation theory in such a way that the embedded renewal process has a finite number of different states (so as to become a Markov renewal process). This extension forms the topics of Section 7.

2. Markov Operators on Vector Lattices

As we hinted in the introduction, formulas of the types (1.3) and (1.4) are meaningful also in the case where λ, λ_n, v, and v_n are elements of an ordered vector space only, rather than measures. Our aim in this section is to make this precise.

For easy reference we shall review some elementary notions concerning vector lattices which will be needed in the sequel. For unexplained concepts we refer to the monograph [19]. (The reader who wants to catch quickly the basic ideas of this paper is advised to think in terms of the forthcoming Example 2.2, i.e., to put $\mathscr{C}$:= the vector lattice of continuous functions having compact support (on an underlying topological space E), $\bar{\mathscr{C}}_+$:= the cone of non-negative lower semicontinuous functions, etc.)

Let $(\mathscr{C}, \geqslant)$ be a *vector lattice* over R, and let $\mathscr{C}_+$ be the cone $\{v \in \mathscr{C};\ v \geqslant 0\}$ of positive elements of $\mathscr{C}$. Let $\mathscr{M}_+$ be a fixed set of positive linear forms on $\mathscr{C}$. For a sequence $v_1 \geqslant v_2 \geqslant \cdots \geqslant 0$ we write $v_n \downarrow 0$ whenever $\lambda(v_n) \downarrow 0$ for all $\lambda \in \mathscr{M}_+$. For two increasing sequences $v = (v_1 \leqslant v_2 \leqslant \cdots)$, $v' = (v'_1 \leqslant v'_2 \leqslant \cdots)$ in $\mathscr{C}_+$ we write $v \geqslant_{\mathscr{M}_+} v'$ (or simply $v \geqslant v'$) if for each m, $(v'_m - v_n)_+ \downarrow 0$ as $n \to \infty$. We say that v and v' are equivalent, and we write $v =_{\mathscr{M}_+} v'$ (or shortly $v = v'$) whenever $v \geqslant v'$ and $v' \geqslant v$. The corresponding equivalence classes form an ordered cone $(\bar{\mathscr{C}}_+, \geqslant)$ which we call simply the *completion* of $\mathscr{C}_+$ (w.r.t. $\mathscr{M}_+$). If we want to emphasize the dependence of the completion on the set $\mathscr{M}_+$ we will use the notation $\bar{\mathscr{C}}_+(\mathscr{M}_+)$.

We embed the original cone $\mathscr{C}_+$ into $\bar{\mathscr{C}}_+$ via the map $v \mapsto (v, v, \ldots)$. Note that for two elements $v, v' \in \mathscr{C}_+$ the notation "$v \geqslant_{\mathscr{M}_+} v'$" means the same as "$\lambda(v' - v)_+ = 0$ for all $\lambda \in \mathscr{M}_+$," whereas "$v =_{\mathscr{M}_+} v'$" means "$\lambda\,|v - v'| = 0$ for all $\lambda \in \mathscr{M}_+$."

If a sequence $v_1 \leqslant v_2 \leqslant \cdots$ belongs to the equivalence class $v \in \bar{\mathscr{C}}_+$ we denote this fact by $v_n \uparrow v$. It is clear from the definition of $\bar{\mathscr{C}}_+$ that every $\lambda \in \mathscr{M}_+$ extends uniquely to a positive linear form from $\bar{\mathscr{C}}_+$ into $\bar{R}_+ := R \cup \{\infty\}$.

For any $\lambda \in \mathscr{M}^+ := \mathscr{M}_+ \setminus \{0\}$ let $\|\cdot\|_\lambda$ denote the seminorm defined by $\|v\|_\lambda = \lambda\,|v|$, and let $\hat{\mathscr{C}}(\lambda)$ be the ordinary completion (i.e., the set of equivalence classes of Cauchy sequences) of $\mathscr{C}$ w.r.t. this seminorm. There exists a natural vector lattice structure in $\hat{\mathscr{C}}(\lambda)$, too.

Next we shall introduce the concept of a Markov operator. Let μ be a fixed element of $\mathscr{M}^+$. A positive linear operator $P\colon \mathscr{C}_+ \ni v \mapsto vP \in \bar{\mathscr{C}}_+$ is called a *Markov operator on the vector lattice* $\mathscr{C}$, if

$$\mu(vP) = \mu(v) \qquad \text{for all } v \in \mathscr{C}_+. \tag{2.1}$$

(Note that in order to define a Markov operator we have to specify the linear form μ w.r.t. which (2.1) holds.) A Markov operator P is called *continuous* (w.r.t. $\mathscr{M}_+$), if

$$\mathscr{C}_+ \ni v_n \downarrow 0 \Rightarrow v_n P \downarrow 0. \tag{2.2}$$

It is clear that a continuous Markov operator extends uniquely to a positive linear operator: $\bar{\mathscr{C}}_+ \to \bar{\mathscr{C}}_+$ and that

$$\bar{\mathscr{C}}_+ \ni v_n \uparrow v \in \bar{\mathscr{C}}_+ \Rightarrow v_n P \uparrow vP.$$

Note also that in the special case where $\mathscr{M}_+ = \{\mu\}$ consists only of the special element μ, the Markov operator P is automatically continuous. Hence it extends uniquely to the completion $\bar{\mathscr{C}}_+(\mu) := \bar{\mathscr{C}}_+(\{\mu\})$ as well as to the Cauchy completion $\hat{\mathscr{C}}(\mu)$.

Markov operators generated by transition probabilities and considered as operators on the vector lattice of finite measures provide a standard example. Although our aim is to consider weaker structures than is the vector lattice of measures, let us for the sake of completeness relate the present definitions to this example.

EXAMPLE 2.1. Let $P = \{P(x, A);\ x \in E,\ A \in \mathscr{E}\}$ be a transition probability on a measurable space $(E, \mathscr{E})$. Setting $\lambda P := \int \lambda(dx)\, P(x, \cdot)$ we see that P acts as a positive linear operator on the vector lattice

$$\mathscr{C} := \text{the set of finite signed measures on } (E, \mathscr{E}).$$

Any f belonging to

$$\mathscr{M}_+ := \text{the set of bounded measurable functions from } E \text{ into } R_+,$$

acts as the positive linear form $\mathscr{C}_+ \ni \lambda \mapsto \int f\, d\lambda \in R_+$. If we let

$$\mu := \text{the constant } 1 \in \mathscr{M}^+,$$

we see that P is a continuous Markov operator on $\mathscr{C}$. Clearly $\|\cdot\|_\mu = \|\cdot\|_1$ is the ordinary total variation norm and $\hat{\mathscr{C}}(\mu) = \hat{\mathscr{C}}(1) = \mathscr{C}$ (by the completeness of the space $\mathscr{C}$).

In Section 6 we shall study the class of co-Feller operators. They are defined as follows.

EXAMPLE 2.2. Let E be a locally compact space having countable base. We equip E with its Borel-σ-algebra $\mathscr{B}$. Let

$\mathscr{C}$:= the set of continuous functions from E into R with compact support.

Any λ belonging to

$$\mathscr{M}_+ := \text{the set of positive Radon measures on } (E, \mathscr{B}),$$

acts as the positive linear form $\mathscr{C}_+ \ni f \mapsto \lambda(f) := \int f \, d\lambda \in R_+$. Clearly

$$\overline{\mathscr{C}}_+ = \overline{\mathscr{C}}_+(\mathscr{M}_+) := \text{the set of lower semicontinuous functions from } E \text{ into } \bar{R}_+,$$

and $v_n \uparrow v$ means the usual pointwise monotone convergence. By Lusin's theorem the Cauchy completion $\hat{\mathscr{C}}(\lambda)$ is equal to

$\mathscr{L}(\lambda)$:= the set of real-valued λ-integrable (measurable w.r.t. the ordinary completion $\hat{\mathscr{B}}(\lambda)$ of the σ-algebra $\mathscr{B}$ w.r.t. λ) functions on E.

Let $\mu \in \mathscr{M}^+$ be a fixed non-zero Radon measure on $(E, \mathscr{B})$. A continuous Markov operator on the vector lattice $\mathscr{C}$ (with the specified linear form μ) will be called in the sequel a *co-Feller operator* (*with reference measure* μ).

It is worth while to recall also the probabilistic interpretation of co-Feller operators. Suppose that a co-Feller operator P is induced by the transition probability function $P(x, A)$, $x \in E$, $A \in \mathscr{B}$, of a Markov chain (X_n), that means, for every $v \in \overline{\mathscr{C}}_+$ with $\mu(v) = 1$, the probability measure $\int v(x)\, P(x, \cdot)\, d\mu(x)$ is absolutely continuous w.r.t. μ, and the corresponding Radon–Nikodym derivative has a lower semicontinuous version ($= vP$). It follows that, if the initial state X_0 has a lower semicontinuous probability density, then so have all the subsequent states X_n, $n = 1, 2, \ldots$.

Let us also briefly discuss the dual concept of a Feller operator:

EXAMPLE 2.3. Let E, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{M}_+$, and $\overline{\mathscr{C}}_+$ be as in the previous example. A positive linear operator $P\colon \overline{\mathscr{C}}_+ \ni f \mapsto Pf \in \overline{\mathscr{C}}_+$ is called a *Feller operator*, if

$$P1 = 1.$$

Note that by Dini's theorem the continuity condition (2.2) is then automatic. We have the probabilistic interpretation

$$Pf(x)=E[f(X_1)\mid X_0=x] \qquad \text{for} \quad x\in E, f\in\bar{\mathscr{C}}_+.$$

Suppose that there exists a Radon measure π on $(E,\mathscr{B})$ which is invariant, that means

$$\pi(Pf)=\pi(f) \qquad \text{for all } f\in\mathscr{C}_+.$$

In this case P can also be regarded as a co-Feller operator on $\mathscr{C}$ (with reference measure π). (In probabilistic terms one has then to consider the dual chain of (X_n).) *Thus all results which hold true for co-Feller operators hold true for Feller operators having an invariant Radon measure.*

In Section 7 we shall study Markov operators which "contract the variation":

EXAMPLE 2.4. Let $E:=[0,1]$, $\mu=m:=$ the Lebesgue measure on $[0,1]$,

$$\mathscr{C}:=\{v;\, v \text{ is a differentiable function on } E \text{ with } v'\in\mathscr{L}(m)\}$$

$$\bigvee_0^1 v:=\int_0^1 |v'(x)|\,dx, \qquad \text{the } \textit{variation} \text{ of } v\in\mathscr{C}.$$

A continuous Markov operator $\mathscr{C}\ni v\mapsto vP\in\mathscr{C}$ is said to *contract the variation*, if there exist an integer $N\geqslant 1$ and constants $\rho<1$ and $C<\infty$ such that

$$\bigvee_0^1 vP^N\leqslant\rho\bigvee_0^1 v+C.$$

(A basic reference concerned with these kind of Markov operators is the paper [10] by Lasota and Yorke. In it the authors concentrate on the special case provided by the Perron–Frobenius operator associated with a strictly expanding, piecewise C^2 transformation on $[0,1]$.)

3. RENEWAL REPRESENTATIONS

In this section we assume that P is an arbitrary Markov operator on a general vector lattice $\mathscr{C}$ with a specified linear form $\mu\in\mathscr{M}^+$. We denote

$$\mathscr{C}_1:=\{v\in\mathscr{C}_+;\, \mu(v)=1\}$$

$$\bar{\mathscr{C}}_1:=\{v\in\bar{\mathscr{C}}_+;\, \mu(v)=1\}.$$

Let w be a fixed element of $\mathscr{C}_1$, and let $\mathbf{u} = (u_n; n = 0, 1, ...)$ be a renewal sequence generated by a probability distribution $\mathbf{b} = (b_n;\ n = 1, 2, ...)$ on $N^+ := \{1, 2, ...\}$, i.e., $b_n \geqslant 0$, $\sum_1^\infty b_n = 1$, and $\mathbf{u}$ is the unique solution of the renewal equations

$$u_0 = 1, \qquad u_n = \mathbf{b} * \mathbf{u}_n \left(:= \sum_{m=1}^{n} b_m u_{n-m} \right), \qquad n = 1, 2, ... \tag{3.1}$$

(see, e.g., [14, Sect. 4.1]).

DEFINITION 3.1. The sequence $w\mathbf{P} := (wP^n;\ n = 0, 1, ...)$ is said to have an *undelayed renewal representation* (*with the aid of the renewal sequence* $\mathbf{u}$), if there exists a sequence $\mathbf{w} = (w_n;\ n = 0, 1, ...) \subset \overline{\mathscr{C}}_+$ such that

$$w\mathbf{P} = \mathbf{u} * \mathbf{w}. \tag{3.2}$$

(Formula (3.2) is a short hand notation for the statement

$$wP^n = \sum_{m=0}^{n} u_{n-m} w_m \qquad \text{for all } n = 0, 1, ...)$$

Clearly $w_0 = w$. Note that, since $\mu(wP^n) = \mu(w) = 1$, we have $\sum_{m=0}^{n} u_{n-m} \mu(w_m) = 1$, whence necessarily

$$\mu(w_m) = B_m := \sum_{n=m+1}^{\infty} b_n, \tag{3.3}$$

or shortly, $\mu(\mathbf{w}) = \mathbf{B}$. Therefore $\mu(w_m) \downarrow 0$ as $m \to \infty$ and

$$b_n = \mu(w_{n-1}) - \mu(w_n).$$

We denote

$$M_b := \sum_1^\infty n b_n = \sum_0^\infty B_n.$$

Let v be an arbitrary element of $\overline{\mathscr{C}}_1$. Suppose that the sequence $w\mathbf{P}$ has the undelayed renewal representation (3.2).

DEFINITION 3.2. The sequence $v\mathbf{P} := (vP^n;\ n = 0, 1, ...)$ has a (*delayed*) *renewal representation* (*with the aid of* $\mathbf{u}$), provided that there exist a sequence $\mathbf{v} = (v_n;\ n = 0, 1, ...) \subset \overline{\mathscr{C}}_+$ and a probability distribution $\mathbf{a}(v) = (a_n(v);\ n = 0, 1, ...) \subset R_+$ on $N := \{0, 1, ...\}$ (both depending on v) such that

$$v\mathbf{P} = \mathbf{v} + \mathbf{a}(v) * \mathbf{u} * \mathbf{w}. \tag{3.4}$$

Note that by (3.2) we have also

$$v\mathbf{P} = \mathbf{v} + \mathbf{a}(v) * w\mathbf{P}. \tag{3.5}$$

Since $\mu(vP^n) = \mu(v) = 1$ it follows that

$$\begin{aligned} \mu(v_n) &= A_n(v) := \sum_{m=n+1}^{\infty} a_m(v) \\ a_n(v) &= \mu(v_{n-1}) - \mu(v_n). \end{aligned} \tag{3.6}$$

In particular, $\mu(v_n) \downarrow 0$ as $n \to \infty$.

We shall denote

$$\bar{\mathscr{R}}_1(w) := \{v \in \bar{\mathscr{C}}_1;\, v\mathbf{P} \text{ has a renewal representation of the form (3.4)}\},$$

$$\mathscr{R}_1(w) := \bar{\mathscr{R}}_1(w) \cap \mathscr{C}_1.$$

The proofs of the following facts are obvious:

PROPOSITION 3.1. (i) *$\mathscr{R}_1(w)$ is a convex set, i.e., closed under finite convex combinations.*

(ii) *$\bar{\mathscr{R}}_1(w)$ is convex, too. Moreover, it is closed under countable convex combinations.*

(iii) *So in particular, if $\mathscr{R}_1(w) = \mathscr{C}_1$, then $\bar{\mathscr{R}}_1(w) = \bar{\mathscr{C}}_1$.*

In the case where $\mathscr{R}_1(w) = \mathscr{C}_1$, or equivalently (by (iii) above), where for every $v \in \bar{\mathscr{C}}_1$, the sequence $v\mathbf{P}$ has a renewal representation, we say that *the Markov operator P has a renewal representation (with the aid of* $\mathbf{u}$).

Note that, by (3.3), the elements $B_m^{-1} w_m \in \bar{\mathscr{C}}_1$ for all m. In fact we have the following result:

LEMMA 3.1. *$B_m^{-1} w_m \in \bar{\mathscr{R}}_1(w)$ for all m.*

Proof. A straightforward calculation shows that for each fixed m,

$$B_m^{-1} w_m P^m = B_m^{-1} w_{m+n} + \sum_{k=1}^{n} B_m^{-1} b_{m+k} w P^{n-k} \qquad \text{for} \quad n = 0, 1, \ldots,$$

which is of the desired form (3.5).

Let

$$\bar{\mathscr{R}}_1^{\min}(w) := \left\{ v \in \bar{\mathscr{C}}_1;\, v \text{ is of the form } v = \sum_0^{\infty} \gamma_m w_m \right.$$
$$\left. \text{for some real numbers } \gamma_0, \gamma_1, \ldots \geqslant 0 \right\}.$$

Note that, by (3.3), the coefficients γ_m necessarily satisfy the condition

$$\sum_0^\infty B_m \gamma_m = 1.$$

Moreover, by (3.2), $wP^n \in \bar{\mathscr{R}}_1^{\min}(w)$ for all n. Note also that $\bar{\mathscr{R}}_1^{\min}(w)$ is closed under countable convex combinations, and that by Propositions 3.1 and 3.2(ii), it is contained in $\bar{\mathscr{R}}_1(w)$.

Summarizing the above we obtain the following proposition.

Proposition 3.2. *The set $\bar{\mathscr{R}}_1^{\min}(w)$ is the minimal convex set which is closed under countable convex combinations and which has the property that for every element v therein, the sequence $v\mathbf{P}$ has a renewal representation.*

One can prove quite easily from the mere existence of an undelayed renewal representation the existence of an invariant element for P. A non-zero element $p \in \bar{\mathscr{C}}_+$ is called *invariant for P*, if

$$pP = p.$$

Recall the definition of $M_b := \sum_1^\infty nb_n \leqslant \infty$.

Theorem 1. *Suppose that P is a continuous Markov operator, and that the sequence $w\mathbf{P}$ has the undelayed renewal representation $w\mathbf{P} = \mathbf{w} * \mathbf{u}$. Then the element*

$$W := \sum_0^\infty w_m \in \bar{\mathscr{C}}_+$$

is invariant for P. Moreover,

$$\mu(W) = M_b. \tag{3.7}$$

Proof. From (3.2) and from the renewal eqs. (3.1) we obtain

$$\begin{aligned} wP^{n+1} &= \sum_{m=0}^{n+1} u_{n+1-m} w_m \\ &= \sum_{m=0}^{n} u_{n-m}(w_{m+1} + b_{m+1} w). \end{aligned}$$

On the other hand

$$wP^{n+1} = \sum_{m=0}^{n} u_{n-m} w_m P.$$

It follows that

$$w_m P = w_{m+1} P + b_{m+1} w \qquad \text{for all } m.$$

Summation over m yields the asserted invariance.

In order to see that $\mu(W) = M_b$ note that the identity

$$\mathbf{B} * \mathbf{u} = \mathbf{1} \ (:= \text{the sequence } (1, 1, \ldots))$$

(see [14, formula (4.6)]) when combined with the renewal representation of $w\mathbf{P}$ leads to the identity

$$\mathbf{B} * w\mathbf{P} = \mathbf{1} * \mathbf{w}.$$

Since $\mu(w\mathbf{P}) = \mathbf{1}$, we obtain

$$\mathbf{B} * \mathbf{1} = \mu(\mathbf{w}) * \mathbf{1},$$

or equivalently,

$$\sum_0^n B_m = \mu\left(\sum_0^n w_m\right) \qquad \text{for all } n.$$

Letting $n \to \infty$ gives the asserted identity (3.7).

We say that $w\mathbf{P}$ has an *undelayed positive renewal representation*, if the embedded renewal sequence $\mathbf{u}$ is positive recurrent, i.e., $\mu(W) = M_b < \infty$. Note that then there exists an invariant element

$$p := M_b^{-1} W \in \mathscr{R}_1^{\min}(w) \tag{3.8}$$

We say that *the Markov operator P has a positive renewal representation,* if it has a renewal representation with the aid of a positive recurrent renewal sequence $\mathbf{u}$.

Later we will prove that then the invariant element p given above has certain minimality and uniqueness properties.

For a Harris recurent Markov operator there exists a natural probabilistic interpretation for the renewal representation in terms of an augmented renewal process. Namely, there exists a bivariate Markov chain (X_n, Y_n) taking values in the enlarged state space $E \times \{0, 1\}$ and having the following properties:

(i) (Y_n) is the indicator sequence of a renewal process (i.e., $Y_n = 1 \Leftrightarrow$ there is a renewal epoch at n), and

(ii) the subset $\alpha := E \times \{1\}$ of the state space is an atom for (X_n, Y_n); i.e., the transition probabilities

$$P\{(X_{n+1}, Y_{n+1}) \in \cdot \mid (X_n, Y_n) = (x, 1)\}, \qquad x \in E,$$

do not depend on x (see [14, Sect. 4.4]).

It is clear that we cannot have the same interpretation for our "abstract" renewal representation, since the property (ii) above forces the Markov operator to be non-singular. (Later in Section 7 we will see that even some deterministic systems can possess a renewal representation in the sense of Definitions 3.1 and 3.2.)

There is, however, a weaker type of interpretation, which we shall now describe:

Suppose that, for some element $v \in \overline{\mathscr{C}}_1$, the sequence $v\mathbf{P}$ has the renewal representation (3.4). Let us denote $\tilde{v}_n := A_n(v)^{-1} v_n$ and $\tilde{w}_n := B_n^{-1} w_n$. Recall that they both belong to $\overline{\mathscr{C}}_1$. Let $(V_n;\ n = 0, 1, \ldots)$ be a $\overline{\mathscr{C}}_1$-valued Markov chain defined as follows. The initial distribution is

$$P\{V_0 = \tilde{w}_0\} = a_0(v), \qquad P\{V_0 = \tilde{v}_0\} = 1 - a_0(v),$$

and the transition probabilities are

$$\begin{aligned}
&P\{V_n = \tilde{w}_0 \mid V_{n-1} = \tilde{v}_{m-1}\} = A_{m-1}(v)^{-1}\, a_m(v) \\
&P\{V_n = \tilde{v}_m \mid V_{n-1} = \tilde{v}_{m-1}\} = A_{m-1}(v)^{-1}\, A_m(v) = 1 - A_{m-1}(v)^{-1}\, a_m(v) \\
&P\{V_n = \tilde{w}_0 \mid V_{n-1} = \tilde{w}_{m-1}\} = B_{m-1}^{-1} b_m \\
&P\{V_n = \tilde{w}_m \mid V_{n-1} = \tilde{w}_{m-1}\} = B_{m-1}^{-1} B_m = 1 - B_{m-1}^{-1} b_m, \qquad n, m \geqslant 1.
\end{aligned} \tag{3.9}$$

Note that, (V_n) has in fact a countable state space, namely $\{\tilde{v}_0, \tilde{v}_1, \ldots\} \cup \{\tilde{w}_0, \tilde{w}_1, \ldots\} \subset \overline{\mathscr{C}}_1$. The former states are transient, while the latter are recurrent. It is an easy exercise to calculate the expectation

$$EV_n = v_n + \mathbf{a}(v) * \mathbf{u} * \mathbf{w}_n. \tag{3.10}$$

But due to the renewal representation, the right hand side equals vP^n and so we obtain the formula

$$EV_n = vP^n. \tag{3.11}$$

Thus our renewal representation means that we can embed our (originally deterministic) sequence (vP^n) in a countable Markov chain so that this sequence becomes the expectation of our Markov chain. (The situation is somewhat analogous to the relation between the solution of an elliptic partial differential equation and the associated diffusion process.)

4. Limit Results

The fundamental limit result which holds true for Harris chains is *Orey's convergence theorem* [16]. For aperiodic Harris chains it states that for all initial distributions λ and λ'

$$\lambda P^n - \lambda' P^n \to 0$$

in total variation norm as $n \to \infty$. Hence, in particular, if P has an invariant probability measure π, then $\lambda P^n \to \pi$.

In this section we shall deal with generalizations of Orey's theorem for Markov operators on vector lattices (having a renewal representation).

Throughout this section P will be a continuous Markov operator on a general vector lattice $\mathscr{C}$ with specified linear form $\mu \in \mathscr{M}^+$. Moreover we assume that w is an element of $\mathscr{C}_1$ such that the sequence $w\mathbf{P}$ has the undelayed renewal representation (3.2).

Recall the definition of the seminorm $\|\cdot\|_\mu = \mu\,|\cdot|$. Also recall the definitions of the convex sets

$$\mathscr{R}_1(w) = \{v \in \mathscr{C}_1;\, v\mathbf{P} \text{ has a renewal representation}\},$$

$$\bar{\mathscr{R}}_1(w) = \{v \in \bar{\mathscr{C}}_1;\, v\mathbf{P} \text{ has a renewal representation}\},$$

and let

$$\hat{\mathscr{R}}_1(w) := \text{the closure of } \mathscr{R}_1(w) \text{ w.r.t. the seminorm } \|\cdot\|_\mu.$$

Clearly $\mathscr{R}_1(w) \subset \bar{\mathscr{R}}_1(w) \subset \hat{\mathscr{R}}_1(w) \subset \hat{\mathscr{C}}_1$.

Before formulating the limit results we shall briefly discuss the concept of *periodicity*. Suppose that the renewal sequence $\mathbf{u}$ has period

$$d := \text{g.c.d.}\{n \geqslant 1;\, u_n > 0\} = \text{g.c.d.}\{n \geqslant 1;\, b_n > 0\}. \tag{4.1}$$

Denoting

$$u_i^{(d)} := u_{id},$$

$$w_i^{(d,r)} := w_{id+r} \qquad \text{for} \quad i \geqslant 0,\, 0 \leqslant r < d,$$

we obtain from (3.2) and (4.1)

$$wP^{id+r} = \sum_{j=0}^{i} u_j^{(d)} w_{i-j}^{(d,r)} = \mathbf{u}^{(d)} * \mathbf{w}_i^{(d,r)},$$

whence, for every fixed $r = 0, 1, \ldots, d-1$, the sequence

$$(wP^{id+r};\, i = 0, 1, \ldots)$$

has an undelayed renewal representation with the aid of the aperiodic renewal sequence $\mathbf{u}^{(d)} = (u_{id};\, i = 0, 1, \ldots)$.

Similarly, the analysis of the delayed periodic case can be reduced to the aperiodic case.

Due to these observations we shall henceforth assume that $d = 1$, i.e., we shall deal with the aperiodic case only.

The following theorem generalizes Orey's convergence theorem:

THEOREM 2. (i) *With the assumptions stated above*

$$\|vP^n - v'P^n\|_{\mu} \to 0 \qquad as \quad n \to \infty \tag{4.2}$$

for all $v, v' \in \hat{\mathscr{R}}_1(w)$.

(ii) *If, in addition,* $\mathbf{u}$ *is positive recurrent, then*

$$\|vP^n - p\| \to 0 \qquad as \quad n \to \infty \tag{4.3}$$

for all $v \in \hat{\mathscr{R}}_1(w)$, *where* p *is the invariant element given by the formula* (3.8). *In fact, for* $v \in \bar{\mathscr{R}}_1(w)$ *a slightly stronger result holds true, namely, there exist sequences* $\mathscr{C}_+ \ni p^{(n)} \uparrow p$ *and* $R_+ \ni \beta^{(n)}(v) \uparrow 1$ *such that*

$$vP^n \geqslant \beta^{(n)}(v)\, p^{(n)} \qquad for\ all\ n. \tag{4.4}$$

Proof. Since P is a linear contraction on $\hat{\mathscr{C}}(\mu)$, it is sufficient to prove the assertion (4.2) for elements $v \in \mathscr{R}_1(w)$, $v' = w$. As in the "renewal proof" of Orey's theorem for Harris chains (see [14, Proof of Theorem 6.7]) we have the inequality

$$\begin{aligned}\|vP^n - wP^n\| &:= \mu\ |vP^n - wP^n| \\ &\leqslant \mu(v_n) + |\mathbf{a}(v) * \mathbf{u} - \mathbf{u}| * \mu(\mathbf{w})_n.\end{aligned}$$

By (3.6) the first term on the right hand side tends to zero. By (3.3) the second term is equal to

$$|\mathbf{a}(v) * \mathbf{u} - \mathbf{u}| * \mathbf{B}_n,$$

and according to Orey's convergence theorem for renewal sequences (see [14, Theorem 6.1]) it tends to zero, too. So we have proved (4.2).

Clearly (4.4) implies (4.3). Thus we are left with the proof of the stronger statement (4.4).

Let $[\alpha] :=$ the integer part of the real number α. We have

$$\begin{aligned}vP^n &\geqslant \mathbf{a}(v) * \mathbf{u} * \mathbf{w}_n \\ &\geqslant \sum_{m=0}^{[n/2]} \mathbf{a}(v) * \mathbf{u}_{n-m} w_m \\ &\geqslant \inf_{i \geqslant [n/2]} \mathbf{a}(v) * \mathbf{u}_i \sum_{m=0}^{[n/2]} w_m.\end{aligned}$$

Let

$$p'_n := M_b^{-1} \sum_{m=0}^{[n/2]} w_m$$

$$\beta^{(n)}(v) := M_b \inf_{i \geqslant [n/2]} \mathbf{a}(v) * \mathbf{u}_i.$$

Now $p'_n \uparrow p$ by the definition of p whereas $\beta^{(n)}(v) \uparrow 1$ by the renewal theorem. Via a diagonal procedure we can find elements $p^{(n)} \leqslant p'_n$, $p^{(n)} \in \mathscr{C}_+$ with the desired property $p^{(n)} \uparrow p$.

COROLLARY 4.1. (i) *If the Markov operator P has a renewal representation, then* (4.2) *holds true for all $v, v' \in \hat{\mathscr{C}}_1$.*

(ii) *If P has a positive renewal representation, then* (4.3) *holds true for all $v \in \hat{\mathscr{C}}_1$ and* (4.4) *holds true for all $v \in \bar{\mathscr{C}}_1$.*

Using the convergence result (4.3) and the inequality (4.4) we can prove a uniqueness result for the invariant elements of P in the case of a positive renewal representation:

COROLLARY 4.2. *Suppose that P has a positive renewal representation.*

(i) *Let $p' \in \hat{\mathscr{C}}_1$ be any subinvariant* (mod μ) *element, that means*

$$\mu(p' - p'P)_+ = 0.$$

Then necessarily

$$p' =_\mu p \qquad (\textit{that means } \mu\,|p' - p| = 0).$$

(ii) *Let $p' \in \bar{\mathscr{C}}_1$ be any (properly) subinvariant element, that means*

$$p'P \leqslant p'.$$

Then necessarily

$$p' \geqslant p \qquad \textit{and} \qquad p' =_\mu p.$$

Proof. (i) Since $\mu(p'P) = \mu(p')$ we have $p'P^n =_\mu p'$. Setting now $v = p'$ in (4.3) yields the desired uniqueness result.

(ii) By the inequality (4.4)

$$p' \geqslant p'P^n \geqslant \beta^{(n)}(p')\, p^{(n)} \uparrow p \qquad \text{as} \quad n \to \infty.$$

Since $\mu(p') = \mu(p)$ it follows that $p' =_\mu p$.

Similarly as Orey's convergence theorem we could extend other limit theorems from Harris chains such as rates of convergence in Orey's theorem, ratio limit theorems, sums of operator iterates, and presumably also central limit and large deviation theorems.

Let us briefly discuss the concept of geometric recurrence and the associated geometric convergence in Orey's theorem. (This concept will appear in Section 7 in the context of Markov operators which contract the variation.)

Recall that a renewal sequence $\mathbf{u}$ is called geometrically recurrent if

$$\sum_{1}^{\infty} b_n r^n < \infty \qquad \text{for some } r > 1.$$

In the aperiodic case it then follows that

$$u_n \to u_\infty \qquad \text{as} \quad n \to \infty$$

with a geometric rate. We shall then call $\mathbf{u}$ a *geometrically ergodic* renewal sequence.

Suppose now that the embedded renewal sequence $\mathbf{u}$ in the renewal representation of $w\mathbf{P}$ is geometrically ergodic. Imitating again the proof for Harris chains (see, e.g., [14, Theorem 6.14]) one could then show that

$$\|wP^n - p\|_\mu \to 0 \qquad \text{with a geometric rate as } n \to \infty,$$

and more generally,

$$\|vP^n - p\|_\mu \to 0 \qquad \text{with a geometric rate as } n \to \infty$$

for all $v \in \bar{\mathscr{R}}_1(w)$ satisfying the condition

$$\sum_{0}^{\infty} a_n(v)\, r^n < \infty \qquad \text{for some } r = r(v) > 1.$$

5. Filling Schemes and Renewal Representations

Let P be a Markov operator on a vector lattice $\mathscr{C}$ with a specified linear form $\mu \in \mathscr{M}^+$. Let us fix an element $w \in \mathscr{C}_1$, and let $v \in \bar{\mathscr{C}}_1$ be arbitrary. (We do not assume having any kind of a renewal representation at the moment.) Consider the following *filling scheme*:

$$\begin{cases} v = a_0 w + v_0 \\ v_0 P = a_1 w + w_1 \\ \cdots \\ v_{n-1} P = a_n w + v_n \\ \cdots, \end{cases} \tag{5.1}$$

where $\mathbf{a} = (a_n;\ n = 0, 1, \ldots) \subset R_+$ and $\mathbf{v} = (v_n;\ n = 0, 1, \ldots) \subset \bar{\mathscr{C}}_+$. Note that, since $\mu(w) = \mu(v) = 1$, it follows that $\sum_1^\infty a_n \leqslant 1$.

We will call the scheme *successful*, provided that $\mathbf{a}$ is a proper probability distribution on N, i.e., $\sum_0^\infty a_n = 1$. Clearly this is equivalent to the statement $\mu(v_n) \downarrow 0$.

The given fixed element w is called *recurrent*, if the element wP has a successful filling scheme, i.e., there exist a probability distribution $\mathbf{b}=(b_n; n=1,2,...)$ on N^+ and a sequence $\mathbf{w}=(w_n; n=0,1,...)\subset\mathscr{C}_+$ with $w_0=w$ such that

$$\begin{cases} wP=b_1w+w_1, \\ w_1P=b_2w+w_2, \\ \cdots \\ w_{n-1}P=b_nw+w_n, \\ \cdots. \end{cases} \tag{5.2}$$

Let $\mathbf{u}=(u_n; n=0,1,...)$ be the renewal sequence generated by the probability distribution $\mathbf{b}$. We have the following simple but important result:

THEOREM 3. (i) *An element $w\in\mathscr{C}_1$ is recurrent (with successful filling scheme* (5.2)) *if and only if the sequence $w\mathbf{P}$ has the undelayed renewal representation* (3.2).

(ii) *Suppose that w is recurrent. For any $v\in\mathscr{C}_1$, v has the successful filling scheme* (5.1) *if and only if $v\mathbf{P}$ has the renewal representation* (3.4) (*with* $\mathbf{a}(v)=\mathbf{a}$).

Proof. Suppose that w is recurrent. Let $\mathbf{u}$ be the renewal sequence generated by $\mathbf{b}$. Then

$$wP=b_1w+w_1=u_1w+u_0w_1,$$

i.e., (3.2) holds true for $n=1$. The general case follows easily by induction: Assuming that $wP^n=\mathbf{u}*\mathbf{w}_n$ we have

$$\begin{aligned} wP^{n+1} &= \sum_{m=0}^{n} u_{n-m}w_mP \\ &= \sum_{m=0}^{n} u_{n-m}(b_{m+1}w+w_{m+1}) \qquad \text{by (5.2)}, \\ &= u_{n+1}w_0+\sum_{m=1}^{n+1} u_{n+1-m}w_m \end{aligned}$$

by the renewal equation and since $w=w_0$. But now we have proved that $wP^{n+1}=\mathbf{u}*\mathbf{w}_{n+1}$.

Suppose, conversely, that $wP^n=\mathbf{u}*\mathbf{w}_n$. By operating from the left by "$\mathbf{B}*$" and recalling that $\mathbf{B}*\mathbf{u}=\mathbf{1}$ we obtain

$$\mathbf{B}*w\mathbf{P}_n=\mathbf{1}*\mathbf{w}_n=\sum_0^n w_m.$$

It follows that

$$\sum_{m=0}^{n-1} w_m P + B_n w = \sum_{m=0}^{n-1} B_{n-1-m} w P^{m+1} + B_n w$$
$$= \mathbf{B} * w\mathbf{P}_n$$
$$= \sum_0^n w_m.$$

This implies

$$w = w_0$$
$$w_{n-1} P = b_n w + w_n \qquad \text{for} \quad n \geqslant 1$$

as asserted.

The proof of part (ii) is similar in character, and therefore we omit it.

Next we shall introduce a "minorization condition" which is sufficient for the existence of a renewal representation:

THEOREM 4. *Suppose that there exist a constant $\alpha > 0$ and an element $w \in \mathscr{C}_1$ such that for all $v \in \mathscr{C}_1$*

$$vP^{m(v)} \geqslant \alpha w \tag{5.3}$$

for some integer $m(v) \geqslant 0$ (depending on v). Then the Markov operator P has a renewal representation.

Proof. Let $v \in \overline{\mathscr{C}}_1$ be arbitrary. Let v' be an element of $\mathscr{C}_1$ such that $\frac{1}{2}v' \leqslant v$. We have

$$vP^{m(v')} \geqslant \tfrac{1}{2}v' P^{m(v')} \geqslant \tfrac{1}{2}\alpha w.$$

Thus we may as well assume that (5.3) holds true for all $v \in \overline{\mathscr{C}}_1$.

Consider now a fixed $v \in \overline{\mathscr{C}}_1$, and let

$$M_1 = m_1 := m(v),$$
$$v_0 := v,\ v_1 := vP, ..., v_{M_1 - 1} := vP^{M_1 - 1}.$$

Then

$$v_{M_1-1} P = vP^{M_1} = \alpha w + v_{M_1},$$

where

$$v_{M_1} := vP^{M_1} - \alpha w \in \overline{\mathscr{C}}_+ .$$

We shall proceed by induction. Suppose that for some i we are given integers $M_j = m_1 + \cdots + m_j$ $(j=1, ..., i)$ and elements $v_0, v_1, ..., v_{M_i} \in \overline{\mathscr{C}}_+$. We define

$$M_{i+1} := M_i + m_{i+1}, \qquad \text{where} \quad m_{i+1} := m(\mu(v_{M_i})^{-1} v_{M_i})$$

$$v_{M_i+1} := v_{M_i} P,\ v_{M_i+2} := v_{M_i} P^2, ..., v_{M_{i+1}-1} := v_{M_i} P^{m_{i+1}-1},$$

$$v_{M_{i+1}} := v_{M_i} P^{m_{i+1}} - \alpha(1-\alpha)^i w.$$

In fact we have now constructed a filling scheme of the form (5.1) with

$$a_n = \begin{cases} \alpha(1-\alpha)^{i-1}, & \text{when} \quad n = M_i \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\sum_0^\infty a_n = 1$, it follows that v has a successful filling scheme. In particular, the element wP has a successful filling scheme, so w is recurrent (Theorem 3(i)). From part (ii) of Theorem 3 it follows that the whole Markov operator P has a renewal representation.

Note that in the case where P has a positive renewal representation the minorization condition (5.3) is also a necessary condition. In order to see this note that by (4.4) we can take $w = \mu(p^{(n_0)})^{-1} p^{(n_0)}$, $\alpha = \frac{1}{2}\mu(p^{(n_0)}) > 0$ for some fixed n_0. Then $m(v) = \inf\{m: \beta^{(m)}(v) \geq \frac{1}{2}\}$.

A subset $\mathscr{G} \subset \overline{\mathscr{C}}_1$ is called a *basis for the cone* $\mathscr{C}_+$, if for every $v \in \mathscr{C}_+$ there exist non-negative real numbers $\gamma_0, \gamma_1, ...$ and elements $g_1, g_2, ... \in \mathscr{G}$ such that

$$\sum_0^\infty \gamma_i g_i = v. \tag{5.4}$$

Note that, necessarily $\sum_0^\infty \gamma_i = \mu(v)$. It is also clear that, if $\mathscr{G}$ is a basis, then every $v \in \overline{\mathscr{C}}_+$ has a representation of the form (5.4).

As a corollary of Theorem 4 we obtain:

COROLLARY 5.1. *Let $\mathscr{G} \subset \overline{\mathscr{C}}_1$ be a basis for the cone $\mathscr{C}_+$. Suppose that there exist a constant $\alpha > 0$ and an element $w \in \mathscr{C}_1$ such that for every $g \in \mathscr{G}$*

$$gP^m \geq \alpha w \qquad \textit{for all } m \geq \textit{some integer } m(g).$$

Then P has a renewal representation.

Proof. Let $v = \sum_0^\infty \gamma_i g_i \in \mathscr{C}_1$ be arbitrary. Let M be such that $\sum_0^M \gamma_i \geq \frac{1}{2}$. We obtain

$$vP^m \geq \tfrac{1}{2}\alpha w \qquad \text{for} \quad m \geq \max_{0 \leq i \leq M} m(g_i),$$

so that (5.3) holds true.

6. Co-Feller Operators

In this section we shall specify our results to the class of co-Feller operators.

Let E be a locally compact space with countable base and let μ be a specified Radon measure (the reference measure) on the Borel-σ-algebra $\mathscr{B}$ of E. Let $\operatorname{Supp}(\mu) := \bigcap\{F;\ F \text{ closed},\ \mu(F^c)=0\}$ be the *support* of μ. *We assume that its interior* $(\operatorname{Supp}(\mu))^0$ *is non-empty.* Recall that P is a co-Feller operator if it acts as a continuous Markov operator on the cone $\overline{\mathscr{C}}_+$ of non-negative lower semicontinuous functions.

First we give a sufficient condition for a co-Feller operator to have a renewal representation:

THEOREM 5. *Let P be a co-Feller operator with reference measure μ, and let $\mathscr{G} \subset \overline{\mathscr{C}}_1$ be a countable basis for the cone $\overline{\mathscr{C}}_+$. Suppose that for some constant $\delta > 0$, some non-empty open set $U \subset \operatorname{Supp}(\mu)$,*

$$\lim_{n\to\infty} \mu\{x \in U;\ gP^n(x) \geqslant \delta\} = \mu(U) \qquad \text{for all } g \in \mathscr{G}. \tag{6.1}$$

Then P has a renewal representation.

Proof. There is no loss of generality to assume that U is relatively compact. Let $g'_n := \min\{gP^n, \delta\}$. Then $g'_n \to \delta$ in μ-measure on U. Let $g_n \in \mathscr{C}_+$, $g_n \leqslant g'_n$ be such that also $g_n \to \delta$ in μ-measure on U. Let $(n') = (n'(g))$ be a subsequence along which $g_{n'} \to \delta$, μ-a.e. on U. By using a diagonal procedure we may assume that the subsequence (n') is the same for all $g \in \mathscr{G}$. Let $V \in \mathscr{B}$ be a set of full measure (i.e., $\mu(V^c)=0$) such that $g_{n'} \to \delta$ on $V \cap U$ for all $g \in \mathscr{G}$. Since the topology of E has a countable basis there is a countable set $D \subset V \cap U$ which is dense in U. For any $x \in D$ we can find a subsequence $(n'') = (n''(x))$ of (n') such that

$$g_{n''}(x) \uparrow \delta \qquad \text{as} \quad n'' \to \infty.$$

By using again the diagonal procedure we can assume that (n'') does not depend on x. Since D is dense in U, it follows that in fact

$$g_{n''} \uparrow \delta \qquad \text{everywhere on the compact set } \bar{U},$$

and now, by Dini's theorem, this convergence is actually uniform. Consequently,

$$gP^{n''} \geqslant g_{n''} \geqslant \tfrac{1}{2}\delta \qquad \text{on } \bar{U} \text{ eventually},$$

which, by the Corollary of Theorem 4, is sufficient for the renewal representation.

Let us formulate Orey's theorem for co-Feller operators.

COROLLARY 6.1. *Suppose that P is a co-Feller operator having an aperiodic renewal representation. Then*

$$vP^n - v'P^n \to 0 \qquad \text{in } \mathscr{L}(\mu)\text{-norm}$$

for all $v, v' \in \mathscr{L}(\mu)$ with $\mu(v) = \mu(v')$. In particular, if P has an invariant element $p \in \mathscr{L}(\mu)$ with $\mu(p) = 1$, then

$$vP^n \to \mu(v)\, p \qquad \text{in } \mathscr{L}(\mu)\text{-norm.}$$

A sufficient condition for the existence of an invariant element $p \in \overline{\mathscr{C}}_1$ is that P has a positive renewal representation. In fact, then there exists a minimal and μ-essentially unique invariant element $p \in \overline{\mathscr{C}}_1$.

It is worth while to formulate these results also in the dual case where P is a Feller operator having an invariant Radon measure π (see Example 2.3). Recall that then P can be viewed also as a co-Feller operator with reference measure π.

For a Feller operator P we shall call a lower semicontinuous function $h \in \overline{\mathscr{C}}_+$ *superharmonic*, if

$$h \geqslant Ph,$$

and *harmonic*, if

$$h = Ph.$$

Note that a superharmonic (resp. harmonic) function is a subinvariant (resp. invariant) element if P is viewed as a co-Feller operator. Hence we obtain immediately the following corollary.

COROLLARY 6.2. *Suppose that P is a Feller operator with an invariant Radon measure π and having an aperiodic positive renewal representation. Then P has a unique minimal π-integrable superharmonic function $h \in \overline{\mathscr{C}}_1$; in fact, h is harmonic. Moreover,*

$$P^n f \to \pi(f) h \qquad \text{in } \mathscr{L}(\pi)\text{-norm as } n \to \infty \text{ for all } f \in \mathscr{L}(\pi).$$

In particular, if π is a probability measure (so that the harmonic function, which is identically equal to 1 belongs to $\overline{\mathscr{C}}_1$) then $h \leqslant 1$ everywhere and $h = 1$, π-almost everywhere. In this case

$$P^n f \to \pi(f) \qquad \text{in } \mathscr{L}(\pi)\text{-norm as } n \to \infty \text{ for all } f \in \mathscr{L}(\pi).$$

As a corollary of Theorem 5 we obtain a result which essentially states that, for co-Feller operators, Orey's convergence theorem and the existence of the renewal representation are equivalent.

COROLLARY 6.3. *Suppose that P is a co-Feller operator having an invariant lower semicontinuous element $p \in \overline{\mathscr{C}}_1$. Suppose that $p(x_0) > 0$ for some $x_0 \in (\mathrm{Supp}(\mu))^0$. Then*

$$vP^n \to p \qquad \textit{in } \mathscr{L}_1(\mu)\textit{-norm for all } v \in \overline{\mathscr{C}}_1, \tag{6.2}$$

if and only if the operator P has a renewal representation.

Proof. We already proved (Theorem 2) that, if P has a renewal representation, then

$$vP^n - v'P^n \to 0 \qquad \text{in } \mathscr{L}(\mu)\text{-norm for all } v, v' \in \mathscr{L}(\mu) \text{ with } \mu(v) = \mu(v').$$

Set $v' = p$ to obtain (6.2).

If, conversely, (6.2) holds then $vP^n \to p$ in μ-measure and hence, by the semicontinuity of p, condition (6.1) of Theorem 5 holds true.

As an illustration we shall study the weak Bernoulli shifts on a finite alphabet:

EXAMPLE 6.1. Let S be a finite set, called the alphabet, and let $\Omega = S^{\times Z}$, $Z := \{..., -1, 0, 1, ...\}$, $\mathscr{F} :=$ the canonical product σ-algebra on Ω. Let X_m denote the coordinate mapping, $X_m(\omega) := \omega_m$ for $\omega \in \Omega$, $m \in Z$, and let

$$\mathscr{F}^0_{-\infty} := \sigma(X_m; m \leqslant 0), \qquad \mathscr{F}^\infty_n := \sigma(X_m; m \geqslant n).$$

The symbol θ denotes the shift operator

$$(\theta\omega)_n := \omega_{n+1}, \qquad n \in Z.$$

Suppose that P is a shift invariant probability measure on $(\Omega, \mathscr{F})$, i.e., under P $(X_n; n \in Z)$ is a stationary sequence of random variables.

Define

$$\begin{aligned} \beta_n &:= 1 - (P \mid \mathscr{F}^0_{-\infty} \vee \mathscr{F}^\infty_n) \wedge (P \mid \mathscr{F}^0_{-\infty} \otimes P \mid \mathscr{F}^\infty_n)(\Omega) \\ &= \tfrac{1}{2} \, \| P \mid \mathscr{F}^0_{-\infty} \vee \mathscr{F}^\infty_n - P \mid \mathscr{F}^0_{-\infty} \otimes P \mid \mathscr{F}^\infty_n \|. \end{aligned}$$

Clearly $\beta_n \downarrow \beta_\infty$ with $0 \leqslant \beta_\infty \leqslant 1$. If $\beta_\infty = 0$, then the shift θ is called *weak Bernoulli.*

With the shift θ there is associated a Markov operator in a natural way: Let π be the probability measure

$$\pi := P(X_0, X_{-1}, ...)^{-1}$$

on $S^{\times Z_-}$ $(Z_- := \{0, -1, ...\})$, and let π_m, $m = 1, 2, ...$, be the marginals

$$\pi_m := P(X_0, ..., X_{-m+1})^{-1}.$$

We set

$$E := \mathrm{Supp}(\pi) = \{x \in S^{\times Z_-};\, \pi_m(x_0, \ldots, x_{-m+1}) > 0 \text{ for all } m\}.$$

On E we have the topology induced by the product topology on $S^{\times Z_-}$, i.e., the basis elements are

$$U_m(x) := \{y \in E;\, y_0 = x_0, \ldots, y_{-m+1} = x_{-m+1}\}, \qquad x \in E, m = 1, 2, \ldots .$$

For each $x_1 \in S$, $x \in E$ let

$$P(x_1 \mid x) := \lim_{m \to \infty} P\{X_1 = x_1 \mid X_0 = x_0, \ldots, X_{-m+1} = x_{-m+1}\}$$

and for any measurable $f := E \to R_+$ define

$$Pf(x) := \sum_{x_1} f(x_1, x_0, \ldots)\, P(x_1 \mid x_0, x_{-1}, \ldots), \qquad x = (x_0, x_{-1}, \ldots) \in E.$$

We assume that for all $x \in E$,

$$\inf_{y \in U_m(x)} P(x_1 \mid y) \uparrow P(x_1 \mid x) \qquad \text{as} \quad m \to \infty. \tag{6.3}$$

This condition makes the operator $f \mapsto Pf$ a Feller operator. Clearly π is invariant for P.

We have the following corollary:

COROLLARY 6.4. *Suppose that the condition* (6.3) *guaranteeing the Feller property is satisfied. Then P has a renewal representation if and only if the shift θ is weak Bernoulli.*

A sufficient condition for this is: For some constant $\delta > 0$ we have

$$\lim_{n \to \infty} P\left\{\frac{dP \mid \mathscr{F}_{-\infty}^{0} \vee \mathscr{F}_{n}^{n+m-1}}{d(P \mid \mathscr{F}_{-\infty}^{0} \otimes P \mid \mathscr{F}_{n}^{n+m-1})} \geqslant \delta\right\} = 1 \qquad \textit{for all } m \geqslant 1. \tag{6.4}$$

Proof. Suppose that θ is weak Bernoulli. Let $g := E^{\times m} \to [-1, 1]$ be measurable and let $g^*(x) = g(x_0, \ldots, x_{-m+1})$, $x \in E$. Since $g(X_{n+m-1}, \ldots, X_n)$ is $\mathscr{F}_n^\infty$-measurable it follows that

$$\begin{aligned} 2\beta_n &\geqslant E\,|E(g(X_{n+m-1}, \ldots, X_n) \mid \mathscr{F}_{-\infty}^{0}) - Eg(X_{n+m-1}, \ldots, X_n)| \\ &= \int_E |P^{n+m-1} g^*(x) - \pi(g^*)|\, \pi(dx). \end{aligned}$$

Since functions g^* of the above form are dense in $\mathscr{L}(\pi)$, it follows that

$$P^n f \to \pi(f) \qquad \text{in } \mathscr{L}(\pi)\text{-norm for all } f \in \mathscr{L}(\pi),$$

i.e., Orey's convergence theorem holds true. But, as we observed in Corollary 6.3 this implies the renewal representation.

Conversely, if we have the renewal representation, then we have Orey's theorem. It is easy to see that then the shift θ is weak Bernoulli.

In order to see that condition (6.4) is sufficient for the weak Bernoulli property, note that it implies

$$\lim_{n \to \infty} \pi\{y \in E;\, P^n f(y) \geqslant \delta\pi(f)\} = 1$$

for all indicator functions

$$f(\cdot) = 1_{U_m(x)}(\cdot), \qquad x \in E,\ m = 1, 2, \ldots .$$

But, when suitably normed, these clearly form a basis for the cone $\mathscr{C}_+$, and thus Theorem 5 can be applied.

7. Markov Renewal Representations

When dealing, e.g., with Markov operators which contract the variation (see Example 2.4) we encounter a situation, which is slightly more general than is the minorization condition (5.3) of Theorem 5. Namely, there will only exist a finite collection $\{w(1), \ldots, w(d)\}$ of elements of $\mathscr{C}_1$ such that

$$vP^m \geqslant \beta w(i) \tag{7.1}$$

for some $i = i(v) \in \{1, \ldots, d\}$, $m = m(v) \geqslant 0$ depending on v, and some $\beta > 0$ (not depending on v).

This condition leads naturally to the concept of a Markov renewal representation for a Markov operator, that means a generalization involving d-dimensional vector valued quantities instead of scalar valued quantities as we had before.

Thus, for example, instead of renewal sequences we have to deal with Markov renewal sequences (with finite state space). These are formally defined as follows:

Let $\mathbf{b} = (b_n;\, n = 1, 2, \ldots)$ be a sequence of elements in $R_+^{d \times d}$:= the set of $d \times d$-matrices with positive elements. Suppose that the sum $B := \sum_1^\infty b_n$ is a stochastic matrix, i.e.,

$$\sum_{j=1}^{d} \sum_{n=1}^{\infty} b_n(i, j) = 1 \qquad \text{for all } i = 1, \ldots, d.$$

Let $\mathbf{u} = (u_n;\ n = 0, 1, \ldots) \subset R_+^{d \times d}$ be the corresponding *Markov renewal sequence*, i.e., the unique solution of the *Markov* renewal equations

$$u_0 = I := \text{the identity matrix},$$

$$u_n = \mathbf{b} * \mathbf{u}_n \left(:= \sum_{m=1}^{n} b_m u_{n-m} \right), \qquad n = 1, 2, \ldots .$$

Let us next describe what we mean by the concept of a Markov renewal representation. Let for a moment P be a general Markov operator on a general vector lattice $\mathscr{C}$ with specified element $\mu \in \mathscr{M}^+$. Let $\mathbf{u}$ be a Markov renewal sequence as above and $w \in \mathscr{C}_1^{d \times 1}$ be a d-dimensional column vector $w = (w(1), ..., w(d))^T$ with elements $w(i) \in \mathscr{C}_1$ $(:= \{v \in \mathscr{C}_+ ;\ \mu(v) = 1\})$. The sequence $w\mathbf{P} := (wP^n;\ n = 0, 1, ...)$ ($wP^n :=$ the column vector $(w(1)\, P^n, ..., w(d)\, P^n)^T$) is said to have an *undelayed Markov renewal representation* (with the aid of the Markov renewal sequence $\mathbf{u}$), if

$$w\mathbf{P} = \mathbf{u} * \mathbf{w} \tag{7.2}$$

for some sequence $\mathbf{w} = (w_n;\ n = 0, 1, ...)$ in $\bar{\mathscr{C}}_+^{d \times 1}$. (A generic element of the sequence $\mathbf{u} * \mathbf{w}$ is given by the column vector $\mathbf{u} * \mathbf{w}_n$ with components

$$\mathbf{u} * \mathbf{w}_n(i) = \sum_{j=1}^{d} \sum_{m=0}^{n} u_{n-m}(i, j)\, w_m(j), \qquad i = 1, ..., d.$$

Suppose now that the sequence $w\mathbf{P}$ has the undelayed representation (7.2). For an arbitrary element $v \in \mathscr{C}_1$ we say that the sequence $v\mathbf{P}$ has a *(delayed) Markov renewal representation*, provided that there exist a sequence $\mathbf{v} = (v_n;\ n = 0, 1, ...) \subset \bar{\mathscr{C}}_1$ and a sequence $\mathbf{a}(v) = (a_n(v);\ n = 0, 1, ...) \subset \bar{\mathscr{C}}_1^{1 \times d}$ (both depending on v) such that $\sum_{j=1}^{d} \sum_{n=0}^{\infty} a_n(j; v) = 1$ and

$$v\mathbf{P} = \mathbf{v} + \mathbf{a}(v) * \mathbf{u} * \mathbf{w} \qquad (= \mathbf{v} + \mathbf{a}(v) * w\mathbf{P} \quad \text{by (7.2)}). \tag{7.3}$$

Similarly as in the "scalar case" we see that the following filling scheme is equivalent to the undelayed Markov renewal representation (7.2),

$$\begin{cases} w = w_0 \\ w_0 P = b_1 w + w_1 \\ \cdots \\ w_{n-1} P = b_n w + w_n \\ \cdots \end{cases} \tag{7.4}$$

with $\sum_1^{\infty} b_n = B =$ a stochastic matrix and $w_0, w_1, ... \in \bar{\mathscr{C}}_1^{d \times 1}$.

Suppose that (7.4) holds true (or equivalently, that $w\mathbf{P}$ has the undelayed Markov renewal representation (7.2)). Then the following filling scheme is equivalent to the Markov renewal representation (7.3),

$$\begin{cases} v = a_0(v) w + v_0 \\ v_0 P = a_1(v) w + v_1 \\ \cdots \\ v_{n-1} P = a_n(v) w + v_n \\ \cdots, \end{cases} \tag{7.5}$$

where $\sum_{j=1}^{d} \sum_{n=0}^{\infty} a_n(j; v) = 1$ and $v_0, v_1, ... \in \bar{\mathscr{C}}_1$.

Let (Z_n) be the embedded Markov chain associated with the Markov renewal sequence $\mathbf{u}$, i.e., (Z_n) is a Markov chain with state space $\{1, ..., d\}$ and transition probabilities

$$B(i, j) = \sum_{n=1}^{\infty} b_n(i, j).$$

Let $\{R^{(1)}, ..., R^{(K)}, T\}$ be the decomposition of the state space $\{1, ..., d\}$ of (Z_n) into K recurrent classes $R^{(1)}, ..., R^{(K)}$ and a transient class T. For each $k = 1, ..., K$ there exists an invariant probability distribution $\pi^{(k)}$ on $\{1, ..., d\}$ concentrated on R_k and such that

$$\pi^{(k)} B = \pi^{(k)}.$$

Note that, since the state space is finite, there always exists at least one recurrent class. If $K = 1$, i.e., there is only one recurrent class, then we shall call the Markov renewal representation *irreducible*.

A recurrent class $R^{(k)}$ is called *positive recurrent for the Markov renewal sequence* $\mathbf{u}$, if

$$M_b^{(k)} := \sum_{n=1}^{\infty} n \sum_{i, j \in R^{(k)}} \pi^{(k)}(i)\, b_n(i, j)$$

is finite.

For each k there are the absorption probabilities $h^{(k)}(i)$, $i = 1, ..., d$. These satisfy the harmonicity conditions

$$Bh^{(k)} = h^{(k)}$$

with the boundary conditions $0 \leqslant h^{(k)}(i) \leqslant 1$, $h^{(k)}(i) = 1$ for $i \in R^{(k)}$, $h^{(k)}(i) = 0$ for $i \in R^{(k')}$ with $k' \neq k$, $\sum_k h^{(k)}(i) = 1$ for every i. Let us denote

$$A(i; v) := \sum_{n=0}^{\infty} a_n(i; v),$$

and by $A^{(k)}(v)$ the "absorption probability from v to the class $R^{(k)}$." The latter is given by

$$A^{(k)}(v) := \sum_{i=1}^{d} A(i; v)\, h^{(k)}(i).$$

Remark. Note that, if $w\mathbf{P}$ has an undelayed Markov renewal representation, then for each state i in any recurrent class $R^{(k)}$, the component sequence $w(i)\mathbf{P}$ has an undelayed renewal representation. Namely,

$$w(i)\mathbf{P} = \mathbf{u}^{(i)} * \mathbf{w}_n^{(i)},$$

where $\mathbf{u}^{(i)}$ denotes the embedded renewal sequence associated with the recurrent state i of the Markov renewal sequence $\mathbf{u}$,

$$\mathbf{w}^{(i)} = \mathbf{b}^{(i)} * \mathbf{w},$$

and $\mathbf{b}^{(i)} = (b_m^{(i)}(j);\ m = 0, 1, ...;\ j = 1, ..., d)$ are the "taboo probabilities" satisfying the equations

$$\mathbf{u}_n(i, j) = \sum_m u_{n-m}^{(i)} b_m^{(i)}(j).$$

Similarly we can see that if $v\mathbf{P}$ has a (delayed) Markov renewal representation then $v\mathbf{P}$ has in fact also a (delayed) renewal representation. These facts will not be exploited in the sequel.

Let us briefly study the invariant elements produced by a Markov renewal representation. Suppose that we have the undelayed representation (7.2) with the aid of a Markov renewal sequence $\mathbf{u}$.

Precisely as in the scalar case (see the proof of Theorem 1) we obtain the equalities

$$w_m P = w_{m+1} + b_{m+1} w \qquad \text{for} \quad m = 0, 1, \dots .$$

Summation over m yields now (we write $W := \sum_0^\infty w_m$; note that $W \in \mathscr{C}_+^{d \times 1}$)

$$WP = \sum_1^\infty w_m + Bw.$$

After recalling that $w_0 = w$ we obtain

$$w + WP = Bw + W.$$

Operating from the left by "$\pi^{(k)}$" leads to

$$\pi^{(k)} WP = \pi^{(k)} W$$

so that $W^{(k)} := \pi^{(k)} W$ is an invariant element for P, for every $k = 1, ..., K$. We have (cf. (3.7))

$$\mu(W^{(k)}) = M_b^{(k)}$$

so that with each positive recurrent (for the Markov renewal sequence $\mathbf{u}$) class $R^{(k)}$ there is associated an invariant element

$$p^{(k)} := (M_b^{(k)})^{-1} W^{(k)}$$

with $\mu(p^{(k)}) = 1$.

Let us formulate Orey's convergence theorem in the case of positive recurrence:

THEOREM 6. *Suppose that all the recurrent classes are aperiodic and positive recurrent for the Markov renewal sequence* **u**. *Then for each* $v \in \hat{\mathscr{C}}_1$

$$vP^n \to \sum_{k=1}^{K} A^{(k)}(v)\, p^{(k)} \qquad \text{as} \quad n \to \infty \text{ in } \|\cdot\|_\mu\text{-norm.}$$

Proof. The proof is an obvious modification of the "scalar case" (Theorem 2).

Theorem 4 generalizes to

THEOREM 7. *Suppose that there exist a constant* $\alpha > 0$ *and elements* $w(1), ..., w(d) \in \overline{\mathscr{C}}_1$, *such that for all* $v \in \mathscr{C}_1$ *there exist* $i = i(v) \in \{1, ..., d\}$ *and an integer* $m = m(v)$ *such that*

$$vP^m \geqslant \alpha w(i). \tag{7.6}$$

Then P *has a Markov renewal representation, that means, for every* $v \in \overline{\mathscr{C}}_1$, v**P** *has a Markov renewal representation with the aid of a Markov renewal sequence* **u**.

Proof. As in the proof of Theorem 4 we may assume that the minorization condition holds true for all $v \in \overline{\mathscr{C}}_1$.

Consider now a fixed $v \in \overline{\mathscr{C}}_1$ and let

$$i_1 = i(v), \qquad M_1 = m(v),$$

$$v_0 = v, \qquad v_1 = vP, ..., v_{M_1 - 1} = vP^{M_1 - 1};$$

then

$$v_{M_1 - 1} P = vP^{m(v)} = \alpha w_{i_1} + v_{M_1},$$

where

$$v_{M_1} := vP^{m(v)} - \alpha w(i_1) \in \overline{\mathscr{C}}_+ .$$

Note that $\mu(v_{M_1}) = 1 - \alpha$.

We shall proceed by induction. Suppose that we are given integers $M_1 < \cdots < M_l$ and indices $i_1, ..., i_l$ and elements $v_0, ..., v_{M_l}$ with $\mu(v_{M_l}) = (1 - \alpha)^l$. We define

$$\begin{aligned} i_{l+1} &:= i(v_{M_l}), \\ M_{l+1} &:= M_l + m(v_{M_l}), \\ v_{M_l + j} &:= v_{M_l} P^j, \qquad j = 1, ..., m(v_{M_l}) - 1, \\ v_{M_{l+1}} &:= v_{M_l} P^{m(v_{M_l})} - \alpha(1 - \alpha)^l\, w_{i_{l+1}} \in \overline{\mathscr{C}}_+ . \end{aligned}$$

So we have produced a filling scheme of the type (7.5) with

$$\begin{aligned}
\alpha_{M_1}(i_1) &= \alpha, \\
\alpha_{M_2}(i_2) &= \alpha(1-\alpha), \\
&\cdots \\
\alpha_{M_l}(i_l) &= \alpha(1-\alpha)^l, \\
\alpha_n(j) &= 0 \qquad \text{otherwise.}
\end{aligned}$$

But this scheme is successful, since

$$\sum_j \sum_n a_n(j; v) = \sum_{l=1}^{\infty} \alpha(1-\alpha)^{l-1} = 1.$$

As an example we have Markov operators which contract the variation. Recall that they are Markov operators which satisfy the condition

$$\bigvee_0^1 vP^N \leqslant \rho \bigvee_0^1 v + C \tag{7.7}$$

for some integer $N \geqslant 1$ and constants $\rho < 1$, $C < \infty$.

THEOREM 8. *Suppose that P contracts the variation. Then P has a Markov renewal representation with the aid of a geometrically recurrent Markov renewal sequence. Hence there exists a finite collection of invariant elements $p^{(1)}, ..., p^{(k)} \in \overline{\mathscr{C}}_1$ for P and numbers $A^{(1)}(v), ..., A^{(k)}(v)$ with $0 \leqslant A^{(k)}(v) \leqslant 1$ such that $\sum_1^k A^{(k)}(v) = 1$ and*

$$vP^n \to v^* := \sum_k A^{(k)}(v)\, p^{(k)} \qquad \textit{in } \mathscr{L}(m)\textit{-norm as } n \to \infty \tag{7.8}$$

for all $v \in \mathscr{L}_+(m)$ with $\int_0^1 v(x)\,dx = 1$. If the variation $\bigvee_0^1 v$ is finite, then the convergence rate is geometric.

Proof. Suppose that M is an integer. It is easy to see that for any $v \in \mathscr{C}_+$ with $\int_0^1 v(x)\,dx = 1$ and $\bigvee_0^1 v \leqslant M$

$$v \geqslant \frac{1}{2} \qquad \text{on} \quad \left[\frac{k}{2M}, \frac{k+1}{2M}\right]$$

for some $k = 0, 1, ..., 2M-1$. Note that by the basic property (7.7) of P

$$\bigvee_0^1 vP^{Nn} \leqslant \rho^n \bigvee_0^1 v + C(1-\rho)^{-1}$$

and hence $m(v)$ is of the order of $\log \bigvee_0^1 v$. Note that

$$\begin{aligned}\bigvee_0^1 v_{M_1} &= \bigvee_0^1 (vP^{m(v)} - \alpha w(i_1)) \\ &\leqslant 2C(1-\rho)^{-1} + \alpha \bigvee_0^1 w(i_1) \\ &\leqslant D, \qquad \text{a finite constant.}\end{aligned}$$

It follows that all the subsequent numbers $m(v_{M_i})$ are of the order $\log D =$ a constant. But this together with the geometric probabilities of "success" in the filling scheme imply the geometric ergodicity of the embedded Markov renewal sequence. Hence, we have also the geometric convergence in (7.8).

Remarks. (i) Since the Perron–Frobenius operator associated with a strictly expanding piecewise C^2 transformation contracts the variation (see [10]) our results apply to this case. In particular *we have an example where a deterministic system has a renewal representation.* Moreover, Theorem 8 shows that the subspace of invariant functions is finite dimensional and we have geometric convergence in (7.8). The existence of an invariant function was also proved in [10]. Li and Yorke [11] proved that there is at most a finite number of them.

(ii) It is clear that the property that the transformation is strictly expanding and piecewise C^2 cannot be a necessary condition for the renewal representation. It would be interesting to know which kind of weaker expansivity properties imposed on a transformation would lead to renewal representation. More specifically: *Is it true that ergodic transformations with positive entropy, i.e., transformations which "expand in the mean" have a renewal representation?*

(iii) Another potentially interesting class of applications might be the Markov operators associated with dynamical particle systems (see, e.g., [12]). These Markov operators are typically singular, whence non-Harris, so that the "old" regeneration techniques do not apply.

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